

Nambu-Type Generalization of the Dirac Equation

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Abstract

Nonlinear generalization of the Dirac equation extending the standard paradigm of nonlinear Hamiltonians is discussed. “Faster-than-light telegraphs” are absent for all theories formulated within the new framework. A new metric for infinite dimensional Lie algebras associated with Lie-Poisson dynamics is introduced.

I. INTRODUCTION

In linear quantum mechanics an algebra of observables is associative and therefore can be naturally related to random variables measured in experiments. In nonlinear quantum mechanics observables (at least Hamiltonians) are represented by nonlinear operators which do not naturally lead to a notion of eigenvalue. The problem is that although it is quite easy to define various nonlinear eigenvalues [1–3], it is difficult to consistently associate with them values of random variables measured in experiments (cf. the discussion of propositions in [4] and problems with probability interpretation discussed in [5,6]). There are several possible ways out of the difficulty. First, one may hope that a consistent theory of measurement will be formulated also for fields evolving nonlinearly. An attempt of Weinberg [2] goes in this direction, but the theory he proposed was based on several arbitrary elements including the (nonrelativistic in nature) assumption that a nonlinear theory may involve nonlinear

Hamiltonians and linear momenta. Second, in the approach of Jordan [7] (similar ideas can be found in [6]) one distinguishes between observables and generators of symmetry transformations (say, Hamiltonian and energy). Unfortunately, in practical applications the idea turns out to possess ambiguities as well (some of them are mentioned in [5] in the context of two-level atoms). The third direction was initiated by Mielnik [8] who proposed to discuss the probability interpretation of generalized theories at the abstract level of convex figures of states. A practical difficulty is that it is very difficult to apply such ideas in concrete situations since we have to know the “shape” of the figure of mixed states. The fourth obvious possibility would be to find a nonlinear quantum mechanics where Hamiltonians and other observables could be simply kept linear.

A general framework which on one hand allows for such generalizations, and on the other contains the Weinberg-type theory as a particular case, is presented in this Letter. We start with rewriting the relativistic density matrix formalism of the Dirac equation in a form of a Nambu-type bracket introduced by Białynicki-Birula and Morrison [16], but here generalized to relativistic and multi-particle systems. Next, we prove that a large class of Nambu-type theories based on the triple bracket is free of the causality problems discussed in the context of the Weinberg theory [20,21,18,14,22,23]. We prove also a theorem stating that at least for a large class of initial conditions a solution of the generalized equation is a density matrix. These theorems generalize earlier results proved by Polchinski and Jordan for Weinberg’s nonlinear quantum mechanics.

The paper is organized as follows. In Sec. II we describe the Hamiltonian formulation of the Dirac equation. Elements of this formulation can be found in literature but in a form which is not very helpful from the point of view of nonlinear Nambu-type generalizations discussed in Sec. IV. Sec. III presents a compact abstract index formalism which exploits formal analogies between density matrices and world-vectors. As a by-product we discuss an interesting metric structure associated with infinite dimensional Lie algebras which can be used in situations where the standard Killing-Cartan finite-dimensional metric does not exist. In Sec. V we prove two important technical lemmas and the theorem on nonexistence

of “faster-than-light telegraphs”. In Sec. VI we discuss the density matrix interpretation of solutions of the Nambu-type solutions of the generalized Dirac equation. Finally, in Sec. VII we discuss examples showing that even keeping Hamiltonians linear one can obtain a nonlinear dynamics.

In a separate paper we shall present an attempt of a physical interpretation of the generators of the Nambu-type dynamics. The interpretation relates the second generator to Rényi’s α -entropies [24] and leads to interesting correlations between linearity of evolution and possibilities of gaining information by a quantum system.

II. HAMILTONIAN FORMULATION OF THE DIRAC EQUATION

The Dirac equation in the Minkowski space representation is ($\hbar = 1$)

$$i\nabla_{AA'}\psi^A = \frac{m}{\sqrt{2}}\xi_{A'}, \quad (1)$$

$$i\nabla^{AA'}\xi_{A'} = \frac{m}{\sqrt{2}}\psi^A \quad (2)$$

where $\nabla_{AA'} = \nabla^a g_{aAA'}$ and $g_{aAA'}$ denote Infeld-van der Waerden tensors [9] (all indices should be understood in the abstract sense of Penrose). To simplify formulas we shall assume that the derivative ∇^a does not contain four-potentials, but the Lie-Poisson and Lie-Nambu algebraic structures we shall derive below would not be changed if we had considered Dirac fields coupled to the Maxwell field (c.f. [6]).

Contracting Eq. (1) with $g_a{}^{BA'}$, Eq. (2) with $g_{aAB'}$, using identities (87), (88) from the Appendix and the (anti-)self-duality properties of the generators we obtain

$$P_a\psi^A = 2 P^{b*}\sigma_{ba}{}^A{}_B\psi^B + \sqrt{2}mg_a{}^{AB'}\xi_{B'}, \quad (3)$$

$$P_a\xi_{A'} = 2 P^{b*}\bar{\sigma}_{baA'}{}^{B'}\xi_{B'} + \sqrt{2}mg_{aBA'}\psi^B, \quad (4)$$

where $P_a = i\nabla_a$. This form of the Dirac equation can be found in [10] and in analogous form in [11] although none of those authors used an explicitly spinor formulation. Let n^a be an arbitrary future-pointing and normalized ($n^a n_a = 1$) timelike world-vector. Denote

$n^a P_a = i n^a \nabla_a = i n \cdot \nabla$. Contracting Eqs. (3), (4) with n^a , then switching from world-vector to spinor indices, using (83), (84) and $n_{AA'} n^{BA'} = \varepsilon_A^B / 2$, we obtain

$$i n \cdot \nabla \psi^A = i n^{BB'} \nabla_{B'}^A \psi_B + \frac{m}{\sqrt{2}} n^{AB'} \xi_{B'}, \quad (5)$$

$$i n \cdot \nabla \xi_{A'} = i n^{BB'} \nabla_{BA'} \xi_{B'} + \frac{m}{\sqrt{2}} n_{BA'} \psi^B, \quad (6)$$

where $n^{AB'} = n^a g_a^{AB'}$. The complex-conjugated equations are

$$-i n \cdot \nabla \bar{\psi}^{A'} = -i n^{BB'} \nabla_B^{A'} \bar{\psi}_{B'} + \frac{m}{\sqrt{2}} n^{BA'} \bar{\xi}_B, \quad (7)$$

$$-i n \cdot \nabla \bar{\xi}_A = -i n^{BB'} \nabla_{AB'} \bar{\xi}_B + \frac{m}{\sqrt{2}} n_{AB'} \bar{\psi}^{B'}. \quad (8)$$

Let $d\mu(n, x) = e_{abcd} n^a dx^b \wedge dx^c \wedge dx^d$ be the measure on the spacelike hyperplane $x^a n_a - \tau = 0$.

Define the norm

$$\| \Psi \|^2 = \int d\mu(n, x) n^{AA'} (\psi_A \bar{\psi}_{A'} + \bar{\xi}_A \xi_{A'}) \quad (9)$$

where

$$\Psi_\alpha = \begin{pmatrix} \psi_A \\ \xi_{A'} \end{pmatrix}. \quad (10)$$

The Hamilton equations equivalent to (5), (6), (7), and (8) can be obtained from the Hamiltonian function

$$\begin{aligned} H &= H[\Psi, \bar{\Psi}] = H[\psi, \bar{\psi}, \xi, \bar{\xi}] \\ &= \int d\mu(n, x) (\bar{\psi}_{A'}, \bar{\xi}_A) \begin{pmatrix} n^{AA'} & 0 \\ 0 & n^{AA'} \end{pmatrix} \begin{pmatrix} i \overrightarrow{\nabla}_{AB'} & -\frac{m}{\sqrt{2}} \varepsilon_{AB} \\ \frac{m}{\sqrt{2}} \varepsilon_{A'B'} & i \overrightarrow{\nabla}_{BA'} \end{pmatrix} \begin{pmatrix} n^{BB'} & 0 \\ 0 & n^{BB'} \end{pmatrix} \begin{pmatrix} \psi_B \\ \xi_{B'} \end{pmatrix} \\ &= \int d\mu(n, x) (\bar{\psi}_{A'}, \bar{\xi}_A) \begin{pmatrix} n^{AA'} & 0 \\ 0 & n^{AA'} \end{pmatrix} \begin{pmatrix} -i \overleftarrow{\nabla}_{AB'} & -\frac{m}{\sqrt{2}} \varepsilon_{AB} \\ \frac{m}{\sqrt{2}} \varepsilon_{A'B'} & -i \overleftarrow{\nabla}_{BA'} \end{pmatrix} \begin{pmatrix} n^{BB'} & 0 \\ 0 & n^{BB'} \end{pmatrix} \begin{pmatrix} \psi_B \\ \xi_{B'} \end{pmatrix}, \end{aligned} \quad (11)$$

where $\overleftarrow{\nabla}$ and $\overrightarrow{\nabla}$ act to the left and to the right, respectively. Let $I_{AA'} = 2n_{AA'}$ and $\omega^{AA'} = n^{AA'}$, and let us denote the directional derivative $n \cdot \nabla$ by a dot. The Dirac equation can be written as classical Hamilton equations.

$$\begin{aligned}
i\dot{\psi}_A &= I_{AA'} \frac{\delta H}{\delta \psi_{A'}}, & i\dot{\xi}_{A'} &= I_{AA'} \frac{\delta H}{\delta \xi_A}, \\
-i\dot{\bar{\psi}}_{A'} &= I_{AA'} \frac{\delta H}{\delta \bar{\psi}_A}, & -i\dot{\bar{\xi}}_A &= I_{AA'} \frac{\delta H}{\delta \bar{\xi}_{A'}},
\end{aligned} \tag{12}$$

or

$$\begin{aligned}
i\omega^{AA'} \dot{\psi}_A &= \frac{\delta H}{\delta \psi_{A'}}, & i\omega^{AA'} \dot{\xi}_{A'} &= \frac{\delta H}{\delta \xi_A}, \\
-i\omega^{AA'} \dot{\bar{\psi}}_{A'} &= \frac{\delta H}{\delta \bar{\psi}_A}, & -i\omega^{AA'} \dot{\bar{\xi}}_A &= \frac{\delta H}{\delta \bar{\xi}_{A'}}.
\end{aligned} \tag{13}$$

III. COMPACT INDEX FORMULATION

In what follows we will need an efficient and compact abstract index formulation of the Hamilton and Poisson equations. To begin with let us denote the continuous spacetime variables “ x ” appearing in the spinor fields $\psi_A = \psi_A(x)$ and $\xi_{A'} = \xi_{A'}(x)$ by lowercase boldface Roman indices: $\psi_A = \psi_A(\mathbf{a})$, $\psi_B = \psi_B(\mathbf{b})$, etc. Second, let us extend the standard Einstein summation convention by assuming that we both sum over repeated spinor indices and integrate with respect to repeated continuous variables (this will simplify formulas by eliminating integrals in the same way the standard convention eliminates sums). Third, the use of Roman letters for both spinor and spacetime indices allows us to finally avoid writing the continuous indices in formulas explicitly. For example, let $\delta(\mathbf{a}, \mathbf{a}')$ denote the delta function on the spacelike hyperplane. Armed with the new convention we can redefine the Poisson tensor and the symplectic form used in the previous section as follows:

$$I_{AA'} = I_{AA'}(\mathbf{a}, \mathbf{a}') = 2n_{AA'} \delta(\mathbf{a}, \mathbf{a}') \tag{14}$$

$$\omega^{AA'} = \omega^{AA'}(\mathbf{a}, \mathbf{a}') = n^{AA'} \delta(\mathbf{a}, \mathbf{a}'), \tag{15}$$

and the norm (9) becomes

$$\| \Psi \|^2 = \omega^{AA'} \left(\psi_A \bar{\psi}_{A'} + \bar{\xi}_A \xi_{A'} \right). \tag{16}$$

Next define bispinor symplectic form and bispinor Poisson tensor by

$$\omega^{\alpha\beta'} = \begin{pmatrix} n^{AB'} \delta(\mathbf{a}, \mathbf{b}') & 0 \\ 0 & n^{BA'} \delta(\mathbf{b}, \mathbf{a}') \end{pmatrix} = \begin{pmatrix} \omega^{AB'} & 0 \\ 0 & \omega^{BA'} \end{pmatrix} \tag{17}$$

$$I_{\alpha\beta'} = \begin{pmatrix} 2n_{AB'}\delta(\mathbf{a}, \mathbf{b}') & 0 \\ 0 & 2n_{BA'}\delta(\mathbf{b}, \mathbf{a}') \end{pmatrix} = \begin{pmatrix} I_{AB'} & 0 \\ 0 & I_{BA'} \end{pmatrix} \quad (18)$$

The use of primed bispinor indices is consistent with

$$\Psi_\alpha = \begin{pmatrix} \psi_A \\ \xi_{A'} \end{pmatrix}, \quad \bar{\Psi}_{\alpha'} = \begin{pmatrix} \bar{\psi}_{A'} \\ \bar{\xi}_A \end{pmatrix}, \quad (19)$$

Now the norm, the Hamiltonian function, and the equations become

$$\|\Psi\|^2 = \omega^{\alpha\alpha'} \Psi_\alpha \bar{\Psi}_{\alpha'}, \quad (20)$$

$$H[\Psi, \bar{\Psi}] = \omega^{\beta\alpha'} \bar{\Psi}_{\alpha'} H_{\beta\gamma'} \Psi_\delta \omega^{\delta\gamma'}, \quad (21)$$

$$i\dot{\Psi}_\beta = H_{\beta\gamma'} \Psi_\delta \omega^{\delta\gamma'}, \quad (22)$$

$$-i\dot{\bar{\Psi}}_{\gamma'} = \omega^{\beta\alpha'} \bar{\Psi}_{\alpha'} H_{\beta\gamma'} \quad (23)$$

$$i\dot{\Psi}_\alpha = I_{\alpha\alpha'} \frac{\delta H}{\delta \bar{\Psi}_{\alpha'}}, \quad (24)$$

$$-i\dot{\bar{\Psi}}_{\alpha'} = I_{\alpha\alpha'} \frac{\delta H}{\delta \Psi_\alpha} \quad (25)$$

$$i\omega^{\alpha\alpha'} \dot{\Psi}_\alpha = \frac{\delta H}{\delta \bar{\Psi}_{\alpha'}}, \quad (26)$$

$$-i\omega^{\alpha\alpha'} \dot{\bar{\Psi}}_{\alpha'} = \frac{\delta H}{\delta \Psi_\alpha}. \quad (27)$$

The next important step follows from the fact that the Hamiltonian function expressed with the help of the pure-state density matrix $\rho_{\alpha\alpha'} = \Psi_\alpha \bar{\Psi}_{\alpha'}$ as

$$H[\rho] = \omega^{\alpha\beta'} \omega^{\beta\alpha'} H_{\alpha\alpha'} \rho_{\beta\beta'} \quad (28)$$

suggests the identification of pairs of primed and unprimed bispinor indices with single lowercase italic Roman indices: $\alpha\alpha' = a$, $\beta\beta' = b$ etc. Introduce now two metric tensors

$$g^{ab} = \omega^{\alpha\beta'} \omega^{\beta\alpha'} \quad (29)$$

$$g_{ab} = I_{\alpha\beta'} I_{\beta\alpha'} \quad (30)$$

satisfying $g^{ab} = g^{ba}$, $g_{ab} = g_{ba}$, $g_{ab}g^{bc} = \delta_a^c$, where $\delta_a^b = \delta_\alpha^\beta \delta_{\alpha'}^{\beta'}$ and

$$\delta_{\alpha}^{\beta} = \begin{pmatrix} \varepsilon_A^B \delta(\mathbf{a}, \mathbf{b}) & 0 \\ 0 & \varepsilon_{A'}^{B'} \delta(\mathbf{a}', \mathbf{b}') \end{pmatrix}, \quad \delta_{\alpha'}^{\beta'} = \begin{pmatrix} \varepsilon_{A'}^{B'} \delta(\mathbf{a}', \mathbf{b}') & 0 \\ 0 & \varepsilon_A^B \delta(\mathbf{a}, \mathbf{b}) \end{pmatrix}. \quad (31)$$

The Hamiltonian function (28) can be rewritten with the help of the metric tensor as

$$H[\rho] = g^{ab} H_a \rho_b. \quad (32)$$

The metric will be shown below to be a natural substitute for the Cartan-Killing metric used in finite-dimensional Lie algebras and is sometimes used implicitly in quantum optics [13]. Useful are also higher-order tensors which can be constructed from the symplectic form and the Poisson tensor. Define

$$g^{a_1 \dots a_n} = \omega^{\alpha_1 \alpha'_n} \omega^{\alpha_2 \alpha'_1} \omega^{\alpha_3 \alpha'_2} \dots \omega^{\alpha_{n-1} \alpha'_{n-2}} \omega^{\alpha_n \alpha'_{n-1}}, \quad (33)$$

$$g_{a_1 \dots a_n} = I_{\alpha_1 \alpha'_n} I_{\alpha_2 \alpha'_1} I_{\alpha_3 \alpha'_2} \dots I_{\alpha_{n-1} \alpha'_{n-2}} I_{\alpha_n \alpha'_{n-1}}. \quad (34)$$

Denote $g^{a_1} = \omega^{a_1}$, $g_{a_1} = I_{a_1}$. Then

$$\text{Tr}(\rho^n) = g^{a_1 \dots a_n} \rho_{a_1} \dots \rho_{a_n} =: C_n[\rho]. \quad (35)$$

To extend the formalism to the case of N free electrons consider an N -particle, totally anti-symmetric state vector $\Psi_{\alpha}^N = \Psi_{\alpha_1 \dots \alpha_N}$. The corresponding N -particle Hamiltonian function is

$$\begin{aligned} H^N = & \bar{\Psi}_{\alpha'_1 \dots \alpha'_N} \omega^{\beta_1 \alpha'_1} \dots \omega^{\beta_N \alpha'_N} \left(H_{\beta_1 \gamma'_1} I_{\beta_2 \gamma'_2} \dots I_{\beta_N \gamma'_N} + \dots \right. \\ & \left. \dots + I_{\beta_1 \gamma'_1} \dots I_{\beta_{N-1} \gamma'_{N-1}} H_{\beta_N \gamma'_N} \right) \omega^{\delta_1 \gamma'_1} \dots \omega^{\delta_N \gamma'_N} \Psi_{\delta_1 \dots \delta_N}. \end{aligned} \quad (36)$$

Define

$$\omega^{N\alpha\alpha'} = \omega^{\alpha_1 \alpha'_1 \dots \alpha_N \alpha'_N} = \omega^{\alpha_1 \alpha'_1} \dots \omega^{\alpha_N \alpha'_N}, \quad (37)$$

$$I^N_{\alpha\alpha'} = I_{\alpha_1 \alpha'_1 \dots \alpha_N \alpha'_N} = I_{\alpha_1 \alpha'_1} \dots I_{\alpha_N \alpha'_N}. \quad (38)$$

The Hamilton equations equivalent to the N -particle Dirac equation are

$$i\dot{\Psi}^N_{\alpha} = I^N_{\alpha\alpha'} \frac{\delta H^N}{\delta \bar{\Psi}^N_{\alpha'}}, \quad (39)$$

$$-i\dot{\bar{\Psi}}^N_{\alpha'} = I^N_{\alpha\alpha'} \frac{\delta H^N}{\delta \Psi^N_{\alpha}} \quad (40)$$

$$i\omega^{N\alpha\alpha'} \dot{\Psi}^N_{\alpha} = \frac{\delta H^N}{\delta \bar{\Psi}^N_{\alpha'}}, \quad (41)$$

$$-i\omega^{N\alpha\alpha'} \dot{\bar{\Psi}}^N_{\alpha'} = \frac{\delta H^N}{\delta \Psi^N_{\alpha}}. \quad (42)$$

The remaining definitions are obtained by substitutions $\omega^{\alpha\alpha'} \rightarrow \omega^{N\alpha\alpha'}$, $I_{\alpha\alpha'} \rightarrow I^N_{\alpha\alpha'}$ in (29), (30), (33), (34), so that

$$g^{Nab} = g^{a_1 b_1} \dots g^{a_N b_N}, \quad (43)$$

$$g^N_{ab} = g_{a_1 b_1} \dots g_{a_N b_N}. \quad (44)$$

IV. LIE-POISSON AND LIE-NAMBU BRACKETS ASSOCIATED WITH THE DIRAC EQUATION

Let $F = F[\Psi, \bar{\Psi}] = F[\rho]$. The Hamilton equations imply the Poisson bracket equations

$$i\dot{F} = I_{\alpha\alpha'} \left(\frac{\delta F}{\delta \Psi_{\alpha}} \frac{\delta H}{\delta \bar{\Psi}_{\alpha'}} - \frac{\delta H}{\delta \Psi_{\alpha}} \frac{\delta F}{\delta \bar{\Psi}_{\alpha'}} \right) \quad (45)$$

$$= I_{\alpha\beta'} \rho_{\beta\alpha'} \left(\frac{\delta F}{\delta \rho_{\alpha\alpha'}} \frac{\delta H}{\delta \rho_{\beta\beta'}} - \frac{\delta H}{\delta \rho_{\alpha\alpha'}} \frac{\delta F}{\delta \rho_{\beta\beta'}} \right). \quad (46)$$

Analogous forms of the Poisson bracket were used in the context of nonrelativistic nonlinear quantum mechanics by Weinberg [2] (RHS of Eq. (45)) and Jordan [14] (RHS of Eq. (46)). The advantage of (46) lies in a possibility of using it for general density matrices (mixed states). The RHS of (46) can be rewritten in a form of a Lie-Poisson bracket

$$\{F, H\} = \rho_a \Omega^a_{bc} \frac{\delta F}{\delta \rho_a} \frac{\delta H}{\delta \rho_b}, \quad (47)$$

where

$$\Omega^a_{bc} = \delta_{\beta'}^{\alpha'} \delta_{\gamma}^{\alpha} I_{\beta\gamma'} - \delta_{\gamma'}^{\alpha'} \delta_{\beta}^{\alpha} I_{\gamma\beta'} \quad (48)$$

satisfy

$$\Omega_{cb}^a = -\Omega_{bc}^a, \quad (49)$$

$$\Omega_{bc}^a \Omega_{de}^c + \Omega_{ec}^a \Omega_{bd}^c + \Omega_{dc}^a \Omega_{eb}^c = 0. \quad (50)$$

and hence are structure constants of an infinite dimensional Lie algebra. For computational reasons it is important to be able to raise and lower indices in the structure constants. The standard Cartan-Killing metric [15] cannot be used in this context because of the infinite dimension of the algebra: $\Omega_{ad}^c \Omega_{bc}^d$ contains expressions such as $\delta(0)$ which are not distributions in the Schwartz sense and such a metric cannot be invertible. The correct metric is given by (29), (30) [12]. We find

$$\Omega_{abc} = g_{ad} \Omega_{bc}^d = I_{\alpha\beta'} I_{\beta\gamma'} I_{\gamma\alpha'} - I_{\alpha\gamma'} I_{\beta\alpha'} I_{\gamma\beta'} \quad (51)$$

$$\Omega^{abc} = g^{bd} g^{ce} \Omega_{de}^a = -\omega^{\alpha\beta'} \omega^{\beta\gamma'} \omega^{\gamma\alpha'} + \omega^{\alpha\gamma'} \omega^{\beta\alpha'} \omega^{\gamma\beta'} \quad (52)$$

The N -particle generalization is obtained by substituting $\omega^{\alpha\alpha'} \rightarrow \omega^{N\alpha\alpha'}$, $I_{\alpha\alpha'} \rightarrow I_{\alpha\alpha'}^N$ in (52), (51). The form (51) can be used to define the Lie-Nambu bracket

$$[F, G, H] = \Omega_{abc} \frac{\delta F}{\delta \rho_a} \frac{\delta G}{\delta \rho_b} \frac{\delta H}{\delta \rho_c}, \quad (53)$$

and its N -particle generalization

$$[F^N, G^N, H^N]^N = \Omega_{abc}^N \frac{\delta F^N}{\delta \rho_a^N} \frac{\delta G^N}{\delta \rho_b^N} \frac{\delta H^N}{\delta \rho_c^N}. \quad (54)$$

An analogous generalized Nambu bracket was discussed in the nonrelativistic context by Białyński-Birula and Morrison [16]. The linear Liouville-von Neumann equation

$$i\dot{\rho}_a = \{\rho_a, H\}, \quad (55)$$

where $H = H[\rho] = g^{ab} H_a \rho_b$, can be written as

$$i\dot{\rho}_a = [\rho_a, H, S] \quad (56)$$

where the “entropy” $S = S[\rho] = g^{ab} \rho_a \rho_b / 2 = C_2[\rho] / 2$ (cf. Eq. (35)). The total antisymmetry of the structure constants implies that S itself commutes with any observable and hence is

a Casimir invariant. This is analogous to the original bracket introduced by Nambu [17] where the structure constants corresponded to the $o(3)$ Lie algebra and the second generator of evolution was the squared angular momentum which is also a Casimir invariant of $o(3)$. The Jordan-Weinberg-type equations discussed in [14] are of the form (55) but involve Hamiltonian functions which can be nonlinear functionals of ρ . The triple bracket equation (56) suggests two additional possibilities of generalizations of the linear equation (55): (a) equations where both H and S are generalized and (b) equations where H is kept linear but $S \neq C_2/2$. Both possibilities are interesting and provide a general nonlinear framework for quantum mechanics which is beyond the standard paradigm of nonlinear Schrödinger equations. The second possibility is interesting as the first general scheme allowing for nonlinear extensions of quantum mechanics which keeps the algebra of observables unchanged (all approaches known to the author of this Letter involve at least nonlinear Hamiltonians).

V. GENERAL PROPERTIES OF THE LIE-NAMBU BRACKET

In this section we shall derive two important properties of the Lie-Nambu bracket which hold independently of the form of H and S . Consider an N -particle density matrix $\rho^N_a = \rho_{a_1 \dots a_N}$. A K -particle subsystem ($K \leq N$) is described by observables of the form

$$F^K = g^{Nab} F_{a_1 \dots a_K} I_{a_{K+1} \dots a_N} \rho_{b_1 \dots b_N} = g^{Kab} F_{a_1 \dots a_K} \rho_{b_1 \dots b_K}, \quad (57)$$

where

$$\begin{aligned} \rho_{b_1 \dots b_K} &= g^{a_{K+1} b_{K+1}} \dots g^{a_K b_N} I_{a_{K+1}} \dots I_{a_N} \rho_{b_1 \dots b_K b_{K+1} \dots b_N} \\ &= \omega^{N-K} \rho_{b_1 \dots b_K b_{K+1} \dots b_N} \end{aligned} \quad (58)$$

is the subsystem's reduced density matrix. Consider now two, M - and $(N - M - K)$ -particle, subsystems which do not overlap (i.e. no particle belongs to both of them). If their reduced density matrices are

$$\rho^I_d = \rho^I_{d_1 \dots d_M} = \rho_{d_1 \dots d_M d_{M+1} \dots d_N} \omega^{d_{M+1} \dots d_N}, \quad (59)$$

$$\rho^{II}_e = \rho^{II}_{e_{M+K+1} \dots e_N} = \omega^{e_1 \dots e_{M+K}} \rho_{e_1 \dots e_{M+K} e_{M+K+1} \dots e_N} \quad (60)$$

then

Lemma 1.

$$[\rho^I_d, \rho^{II}_e, \cdot]^N = 0. \quad (61)$$

Proof: Reduced density matrices satisfy

$$\frac{\delta \rho^I_d}{\delta \rho_{a_1 \dots a_N}} = \delta_{d_1}^{a_1} \dots \delta_{d_M}^{a_M} \omega^{a_{M+1} \dots a_N}, \quad \frac{\delta \rho^{II}_e}{\delta \rho_{b_1 \dots b_N}} = \omega^{b_1 \dots b_{M+K}} \delta_{e_{M+K+1}}^{b_{M+K+1}} \dots \delta_{e_N}^{b_N}.$$

Consider the expression

$$\begin{aligned} X_c &= \Omega^N \frac{\delta \rho^I_d}{\delta \rho^N_a} \frac{\delta \rho^{II}_e}{\delta \rho^N_b} \\ &= \Omega^N \omega^{a_1 \dots a_M a_{M+1} \dots a_N, b_1 \dots b_{M+K} b_{M+K+1} \dots b_N, c_1 \dots c_N} \delta_{d_1}^{a_1} \dots \delta_{d_M}^{a_M} \omega^{a_{M+1} \dots a_N} \omega^{b_1 \dots b_{M+K}} \delta_{e_{M+K+1}}^{b_{M+K+1}} \dots \delta_{e_N}^{b_N} \\ &= \Omega^N \omega^{a_1 \dots a_M a_{M+1} \dots a_N, b_1 \dots b_{M+K} e_{M+K+1} \dots e_N, c_1 \dots c_N} \omega^{a_{M+1} \dots a_N} \omega^{b_1 \dots b_{M+K}} \\ &= \left(I_{\delta_1 \beta'_1} \dots I_{\delta_M \beta'_M} I_{\alpha_{M+1} \beta'_{M+1}} \dots I_{\alpha_{M+K} \beta'_{M+K}} I_{\alpha_{M+K+1} \varepsilon'_{M+K+1}} \dots I_{\alpha_N \beta'_N} \right. \\ &\quad \times I_{\beta_1 \gamma'_1} \dots I_{\beta_{M+K} \gamma'_{M+K}} I_{\varepsilon_{M+K+1} \gamma'_{M+K+1}} \dots I_{\varepsilon_N \gamma'_N} \\ &\quad \times I_{\gamma_1 \delta'_1} \dots I_{\gamma_M \delta'_M} I_{\gamma_{M+1} \alpha'_{M+1}} \dots I_{\gamma_N \alpha'_N} \\ &\quad - I_{\delta_1 \gamma'_1} \dots I_{\delta_M \gamma'_M} I_{\alpha_{M+1} \gamma'_{M+1}} \dots I_{\alpha_N \gamma'_N} \\ &\quad \times I_{\beta_1 \delta'_1} \dots I_{\beta_M \delta'_M} I_{\beta_{M+1} \alpha'_{M+1}} \dots I_{\beta_{M+K} \alpha'_{M+K}} I_{\varepsilon_{M+K+1} \alpha'_{M+K+1}} \dots I_{\beta_N \alpha'_N} \\ &\quad \times I_{\gamma_1 \beta'_1} \dots I_{\gamma_{M+K} \beta'_{M+K}} I_{\gamma_{M+K+1} \varepsilon'_{M+K+1}} \dots I_{\gamma_N \varepsilon'_N} \left. \right) \\ &\quad \times \omega^{\alpha_{M+1} \alpha'_{M+1}} \dots \omega^{\alpha_N \alpha'_N} \omega^{\beta_1 \beta'_1} \dots \omega^{\beta_{M+K} \beta'_{M+K}} \\ &= I_{\delta_1 \beta'_1} \dots I_{\delta_M \beta'_M} \delta_{\beta'_{M+1}}^{\alpha'_{M+1}} \dots \delta_{\beta'_{M+K}}^{\alpha'_{M+K}} \delta_{\varepsilon'_{M+K+1}}^{\alpha'_{M+K+1}} \dots \delta_{\beta'_N}^{\alpha'_N} \\ &\quad \times \delta_{\gamma'_1}^{\beta'_1} \dots \delta_{\gamma'_{M+K}}^{\beta'_{M+K}} I_{\varepsilon_{M+K+1} \gamma'_{M+K+1}} \dots I_{\varepsilon_N \gamma'_N} \\ &\quad \times I_{\gamma_1 \delta'_1} \dots I_{\gamma_M \delta'_M} I_{\gamma_{M+1} \alpha'_{M+1}} \dots I_{\gamma_N \alpha'_N} \\ &\quad - I_{\delta_1 \gamma'_1} \dots I_{\delta_M \gamma'_M} \delta_{\gamma'_{M+1}}^{\alpha'_{M+1}} \dots \delta_{\gamma'_N}^{\alpha'_N} \\ &\quad \times \delta_{\delta'_1}^{\beta'_1} \dots \delta_{\delta'_M}^{\beta'_M} \delta_{\alpha'_{M+1}}^{\beta'_{M+1}} \dots \delta_{\alpha'_{M+K}}^{\beta'_{M+K}} I_{\varepsilon_{M+K+1} \alpha'_{M+K+1}} \dots I_{\beta_N \alpha'_N} \\ &\quad \times I_{\gamma_1 \beta'_1} \dots I_{\gamma_{M+K} \beta'_{M+K}} I_{\gamma_{M+K+1} \varepsilon'_{M+K+1}} \dots I_{\gamma_N \varepsilon'_N} = 0. \quad (62) \end{aligned}$$

The proof is completed by $[\rho^I_d, \rho^{II}_e, \cdot]^N = X_c \delta / \delta \rho_c = 0$. \square

A straightforward consequence of Lemma 1 is the following important theorem about nonexistence of “faster-than-light telegraphs” for all Nambu-type generalizations of the Dirac equation.

Theorem 2. Consider two, in general nonlinear, observables $F^I[\rho] = F^I[\rho^I]$, $G^{II}[\rho] = G^{II}[\rho^{II}]$ corresponding to two nonoverlapping, M - and $(N - M - K)$ -particle subsystems of a larger N -particle system. Then

$$[F^I, G^{II}, \cdot]^N = 0. \quad (63)$$

Proof:

$$[F^I, G^{II}, \cdot]^N = \Omega^N \frac{\delta \rho_d^I}{\delta \rho_a^N} \frac{\delta \rho_e^{II}}{\delta \rho_b^N} \frac{\delta F^I}{\delta \rho_d^I} \frac{\delta G^{II}}{\delta \rho_e^{II}} \frac{\delta}{\delta \rho_c} = 0. \quad (64)$$

□

The meaning of Theorem 2 is the following. Consider two noninteracting subsystems described by a (possibly nonlinear) Hamiltonian function

$$H[\rho] = H^I[\rho^I] + H^{II}[\rho^{II}]. \quad (65)$$

Then, for *any* S

$$i\dot{F}^I = [F^I, H, S] = [F^I, H^I, S] \quad (66)$$

and the dynamics of a subsystem is generated by the Hamiltonian function of this subsystem. Theorem 2 is a generalization of the theorems of Polchinski [18], which was demonstrated for pure-state density matrices, and Jordan [14] which was formulated in terms of arbitrary density matrices. The results of Polchinski and Jordan referred to Weinberg’s nonlinear quantum mechanics, which is a particular case of our triple-bracket formulation obtained if we put $S = C_2/2$ (in our case S is arbitrary). In addition all those formulations were two-particle and nonrelativistic. Our theorem is the first step towards Fock space and, in larger perspective, relativistic field-theoretic generalization. This result is also interesting in the context of Weinberg’s remark that he “could not find any way to extend the nonlinear

version of quantum mechanics to theories based on Einstein's special theory of relativity" [19].

Lemma 3.

$$[C_n, C_m, \cdot]^N = 0. \quad (67)$$

Proof: To simplify notation we shall not explicitly write the number-of-particles index N in formulas. The indices a, b, c, a_k, b_l are themselves N -particle indices. We have

$$\frac{\delta C_n}{\delta \rho_a} = n g^{b_1 \dots b_{n-1} a} \rho_{b_1} \dots \rho_{b_{n-1}}. \quad (68)$$

Consider the expression

$$\begin{aligned} Y_c^{m,n} &= \Omega_{abc} g^{a_1 \dots a_n a} g^{b_1 \dots b_m b} \rho_{a_1} \dots \rho_{a_n} \rho_{b_1} \dots \rho_{b_m} \\ &= (I_{\alpha\beta'} I_{\beta\gamma'} I_{\gamma\alpha'} - I_{\alpha\gamma'} I_{\beta\alpha'} I_{\gamma\beta'}) \omega^{\alpha_1 \alpha'_1} \omega^{\alpha_2 \alpha'_1} \dots \omega^{\alpha_n \alpha'_{n-1}} \omega^{\alpha_n \alpha'_n} \omega^{\beta_1 \beta'_1} \omega^{\beta_2 \beta'_1} \dots \omega^{\beta_m \beta'_{m-1}} \omega^{\beta_m \beta'_m} \\ &\quad \times \rho_{a_1} \dots \rho_{a_n} \rho_{b_1} \dots \rho_{b_m} \\ &= \delta_{\gamma}^{\alpha_1} \omega^{\alpha_2 \alpha'_1} \omega^{\alpha_3 \alpha'_2} \dots \omega^{\alpha_n \alpha'_{n-1}} \omega^{\beta_1 \alpha'_n} \omega^{\beta_2 \beta'_1} \dots \omega^{\beta_m \beta'_{m-1}} \delta_{\gamma'}^{\beta'_m} \rho_{\alpha_1 \alpha'_1} \dots \rho_{\alpha_n \alpha'_n} \rho_{\beta_1 \beta'_1} \dots \rho_{\beta_m \beta'_m} \\ &\quad - \delta_{\gamma}^{\beta_1} \omega^{\beta_2 \beta'_1} \omega^{\beta_3 \beta'_2} \dots \omega^{\beta_m \beta'_{m-1}} \omega^{\alpha_1 \beta'_m} \omega^{\alpha_2 \alpha'_1} \dots \omega^{\alpha_n \alpha'_{n-1}} \delta_{\gamma'}^{\alpha'_m} \rho_{\alpha_1 \alpha'_1} \dots \rho_{\alpha_n \alpha'_n} \rho_{\beta_1 \beta'_1} \dots \rho_{\beta_m \beta'_m} \\ &= \omega^{\alpha_2 \alpha'_1} \dots \omega^{\alpha_n \alpha'_{n-1}} \omega^{\alpha_{n+1} \alpha'_n} \omega^{\alpha_{n+2} \alpha'_{n+1}} \dots \omega^{\alpha_{n+m} \alpha'_{n+m-1}} \rho_{\gamma \alpha'_1} \dots \rho_{\alpha_{n+m} \gamma'} \quad (69) \end{aligned}$$

$$- \omega^{\beta_2 \beta'_1} \dots \omega^{\beta_m \beta'_{m-1}} \omega^{\beta_{m+1} \beta'_m} \omega^{\beta_{m+2} \beta'_{m+1}} \dots \omega^{\beta_{n+m} \beta'_{n+m-1}} \rho_{\gamma \beta'_1} \dots \rho_{\beta_{n+m} \gamma'} = 0. \quad (70)$$

where we have renamed indices: $(\beta_1, \dots, \beta_m) \rightarrow (\alpha_{n+1}, \dots, \alpha_{n+m})$ in (69), and $(\alpha_1, \dots, \alpha_n) \rightarrow (\beta_{m+1}, \dots, \beta_{n+m})$ in (70). Now

$$[C_m, C_n, \cdot]^N = mn Y_c^{m-1, n-1} \frac{\delta}{\delta \rho_c} = 0$$

which completes the proof. \square

The consequence of Lemma 3 is

Theorem 4. Let $S = S(C_1, \dots, C_k, \dots)$ be any differentiable function of C_1, \dots . Then

$$[C_n, \cdot, S]^N = 0. \quad (71)$$

Proof:

$$[C_n, \cdot, S]^N = \sum_k [C_n, \cdot, C_k]^N \frac{\partial S}{\partial C_k} = 0.$$

□

As a consequence, C_n are constants of motion for all (in general nonlinear) Hamiltonian functions H if the generalized entropies depend on ρ_a via C_k . C_n are, for such a class of entropies, “Casimir invariants” of the triple bracket algebra of observables. These results generalize the theorem of Jordan [14] who proved this property of C_n for $S = C_2/2$ in nonrelativistic nonlinear quantum mechanics.

VI. DENSITY MATRIX INTERPRETATION OF SOLUTIONS OF THE GENERALIZED DIRAC EQUATION

The fact that $C_n = \text{Tr}(\rho^n)$ are constants of motion for any H if S depends on ρ_a via C_k , $k = 1, 2, \dots$, can be used to prove the following

Theorem 5. Let $t \mapsto \rho_a(t)$ be a Hermitian solution of

$$i\dot{\rho}_a = [\rho_a, H, S] \tag{72}$$

where H is arbitrary and $S = S(C_1, \dots, C_k, \dots)$, such that $\rho_a(0)$ is a density matrix having a finite number of nonvanishing eigenvalues $p_k(0)$. Then eigenvalues $p_k(t)$ of $\rho_a(t)$ are constants of motion, i.e. $p_k(t) = p_k(0)$ for $t > 0$.

Proof: Since the nonvanishing eigenvalues of ρ_0 satisfy $0 < p_k(0) \leq 1 < 2$, it follows that for any α $p_k(0)^\alpha$ can be written in the form of a convergent Taylor series. By virtue of the spectral theorem the same is true for ρ_0^α and $\text{Tr}(\rho_0^\alpha)$. Each term of the Taylor expansion of $\text{Tr}(\rho_0^\alpha)$ is proportional to $f_n[\rho_0]$, for some n . But $f_n[\rho_0] = f_n[\rho_t]$ hence

$$\text{Tr}(\rho_0^\alpha) = \text{Tr}(\rho_t^\alpha) = \sum_k p_k(0)^\alpha = \sum_k p_k(t)^\alpha \tag{73}$$

for all real α . Since all $p_k(0)$ are assumed to be known (the initial condition), we know also $\sum_k p_k(0)^\alpha = \sum_k p_k(t)^\alpha$ for any α . We can now apply the result from information theory [24] that the knowledge of

$$\sum_{k=1}^{n<\infty} p_k(t)^\alpha \quad (74)$$

for all α *uniquely* determines $p_k(t)$. The continuity in t implies that $p_k(t) = p_k(0)$. \square

Remark: The assumption that initially the density matrix has a *finite* number of non-vanishing eigenvalues $p_k(0)$ is necessary since the theorem we use in the proof is formulated in [24] for sums (74) with finite n . This result is not fully satisfactory, but is sufficient at least “for all practical purposes”.

VII. TWO EXAMPLES

Consider now the generalization where observables are linear and, in addition, a multiplication of a solution by a number is a symmetry of the evolution equation. To maintain the latter a homogeneity of S must be the same as this of C_2 . An example of homogeneity preserving generalization of $S = C_2/2$ is

$$S_\alpha[\rho] = \left(1 - \frac{1}{\alpha}\right) \frac{(\text{Tr } (\rho^\alpha))^{1/(\alpha-1)}}{(\text{Tr } \rho)^{1/(\alpha-1)-1}} \quad (75)$$

where ρ is a density matrix. The choice of the denominator is important only from the point of view of the homogeneity of the evolution equation. The multiplier $1 - 1/\alpha$ guarantees that the evolution of pure states is the same, and therefore *linear*, for all α . The generalized Liouville-von Neumann equation following from (75) is

$$i\dot{\rho} = \frac{(\text{Tr } (\rho^\alpha))^{1/(\alpha-1)-1}}{(\text{Tr } \rho)^{1/(\alpha-1)-1}} [\hat{H}, \rho^{\alpha-1}]. \quad (76)$$

For pure states and $\text{Tr } \rho = 1$, $\rho^n = \rho$ and the equation reduces to the ordinary, linear one; for mixed states the evolution is nonlinear unless the states are mixed to the extent that ρ is proportional to the unit operator.

The evolution of the (linear) observables is governed by

$$i\dot{F} = \frac{(\text{Tr}(\rho^\alpha))^{1/(\alpha-1)-1}}{(\text{Tr} \rho)^{1/(\alpha-1)-1}} \text{Tr}(\rho^{\alpha-1}[\hat{F}, \hat{H}]) \quad (77)$$

which shows that for the generalized S the time derivative of an observable is not linear in the density matrix. For $\alpha = 2$ the equations reduce to the ordinary linear equations.

Also of interest is the following choice of S_α

$$S_\alpha[\rho] = \frac{1}{2} \frac{(\text{Tr}(\rho^\alpha))^{1/(\alpha-1)}}{(\text{Tr} \rho)^{1/(\alpha-1)-1}}. \quad (78)$$

For pure states the expression reduces to the linear form $\frac{1}{2}\langle\psi|\psi\rangle^2 = \frac{1}{2}\text{Tr}(\rho^2)$, and the density matrix then satisfies

$$i\dot{\rho} = \frac{1}{2} \frac{\alpha}{\alpha-1} \frac{(\text{Tr}(\rho^\alpha))^{1/(\alpha-1)-1}}{(\text{Tr} \rho)^{1/(\alpha-1)-1}} [\hat{H}, \rho^{\alpha-1}] \quad (79)$$

which for pure states and normalized ρ becomes

$$2\frac{\alpha-1}{\alpha}i\dot{\rho} = [\hat{H}, \rho]. \quad (80)$$

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VIII. APPENDIX: INFELD-VAN DER WAERDEN TENSORS AND GENERATORS OF (1/2,0) AND (0,1/2)

This Appendix explains spinor conventions used in this Letter Consider representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of an element $\omega \in SL(2, C)$: $e^{\frac{i}{2}\omega^{ab}\sigma_{ab}}$ and $e^{\frac{i}{2}\omega^{ab}\bar{\sigma}_{ab}}$. The explicit form of the

generators in terms of Infeld-van der Waerden tensors is

$$\frac{1}{2i} \left(g^a_{XA'} g^{bYA'} - g^b_{XA'} g^{aYA'} \right) = \sigma^{ab}{}_X{}^Y, \quad (81)$$

$$\frac{1}{2i} \left(g^a_{AX'} g^{bAY'} - g^b_{AX'} g^{aAY'} \right) = \bar{\sigma}^{ab}{}_{X'}{}^{Y'}. \quad (82)$$

Their purely spinor form is

$$\sigma_{AA'BB'XY} = \frac{1}{2i} \varepsilon_{A'B'} (\varepsilon_{AX} \varepsilon_{BY} + \varepsilon_{BX} \varepsilon_{AY}), \quad (83)$$

$$\bar{\sigma}_{AA'BB'X'Y'} = \frac{1}{2i} \varepsilon_{AB} (\varepsilon_{A'X'} \varepsilon_{B'Y'} + \varepsilon_{B'X'} \varepsilon_{A'Y'}), \quad (84)$$

Dual tensors are ${}^* \bar{\sigma}^{ab}{}_{X'}{}^{Y'} = +i \bar{\sigma}^{ab}{}_{X'}{}^{Y'}$ and ${}^* \sigma^{ab}{}_X{}^Y = -i \sigma^{ab}{}_X{}^Y$.

Additionally the Infeld-van der Waerden tensors satisfy

$$g^a_{XA'} g^{bYA'} + g^b_{XA'} g^{aYA'} = g^{ab} \varepsilon_X{}^Y \quad (85)$$

$$g^a_{AX'} g^{bAY'} + g^b_{AX'} g^{aAY'} = g^{ab} \varepsilon_{X'}{}^{Y'} \quad (86)$$

These equations lead to the identities

$$g^a_{XA'} g^{bYA'} = \frac{1}{2} g^{ab} \varepsilon_X{}^Y + i \sigma^{ab}{}_X{}^Y \quad (87)$$

$$g^a_{AX'} g^{bAY'} = \frac{1}{2} g^{ab} \varepsilon_{X'}{}^{Y'} + i \bar{\sigma}^{ab}{}_{X'}{}^{Y'} \quad (88)$$

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